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Non-Technical Summary

Many countries operate large social security systems. Social security can increase economic efficiency and provide insurance against household risks for which no private markets exist. However, social security systems also impose costs by distorting prices and decisions. The question arises whether the benefits of social security outweigh the costs.

To study the welfare consequences of introducing a social security system this paper develops an analytically tractable model with two overlapping generations where households face many different sources of risks over their life-cycle. Key risks for the analysis of social security are idiosyncratic risks – e.g., the risk of unemployment – and aggregate risks – i.e., the risk of aggregate productivity losses which leads to reductions of wages and asset returns. Our analysis on the welfare effects of social security in light of these risks differs from the previous literature in that, thus far, prior studies have only considered models with one type of risk in isolation. One strand of the literature has examined social security when only aggregate risk is present. The other strand has only considered idiosyncratic risk.

In our model we find that the two risks interact in multiple ways and influence the welfare effects of introducing a social security system. As a result, we show that these effects are less strong on the welfare costs of crowding out than on the welfare gains from insurance. Hence, the interactions of risks increase the total welfare benefits of social security.

The findings of this paper must be investigated in a more realistic quantitative model which is suitably calibrated. This is done in our companion paper, Harenberg and Ludwig (2014). There we document that indeed the interactions of risks overturn conventional findings on the welfare effects of social security and we conclude that the introduction of a minimum flat pension is welfare improving once all household risks are appropriately taken into account.

Social Security in an Analytically Tractable Overlapping Generations Model with Aggregate and Idiosyncratic Risk*

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Abstract

When markets are incomplete, social security can partially insure against idiosyncratic and aggregate risks. We incorporate both risks into an analytically tractable model with two overlapping generations. We derive the equilibrium dynamics in closed form and show that joint presence of both risks leads to overproportional risk exposure for households. This implies that the whole benefit from insurance through social security is greater than the sum of the benefits from insurance against each of the two risks in isolation. We measure this through interaction effects which appear even though the two risks are orthogonal by construction. While the interactions unambiguously increase the welfare benefits from insurance, they can in- or decrease the welfare costs from crowding out of capital formation. The net effect depends on the relative strengths of the opposing forces.

JEL classification: C68; E27; E62; G12; H55

Keywords: social security; idiosyncratic risk; aggregate risk; welfare; insurance; crowding out

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1 Introduction

Almost all industrialized countries have large public social security systems with sizeable pay-as-you-go (PAYG) components. In such systems payments to current pensioners are financed by taxing current workers. Social security can hence improve intergenerational risk sharing by pooling aggregate risks across generations. In addition, most systems have some form of redistributional component. Hence, social security can also insure against idiosyncratic earnings risks for which private markets do not exist and for which other government transfers only provide partial insurance. However, these systems are financed by distortionary taxes. The question arises whether the benefits from insurance outweigh the costs of distortionary taxation.

The present paper demonstrates that the benefits from insurance have been underestimated in the previous literature because either aggregate or idiosyncratic risks have been studied in isolation.[1] We argue that simply combining the findings from this previous literature leads to (potentially) severe biases in the welfare assessments of social security. The reason is that social security can insure both risks and that the benefit from such joint insurance is a convex function of total risk. Hence the whole insurance gain is greater than the sum of the numbers reported in the previous literature. However, this statement only refers to the gains from insurance and is agnostic about the welfare losses of distortionary taxation. If these welfare losses also increase more than additively in both risks, then it is unclear which effect dominates. The objective of the present paper is therefore to characterize the net welfare effects of introducing social security once the benefits from insurance and the welfare losses from distortionary taxation are appropriately taken into account in presence of both risks.

To this aim we develop an overlapping generations model with incomplete markets and a social security system. For reasons of analytical tractability, we assume that a household lives for two periods, so that at each point in time, two generations are simultaneously alive. In the first period of life, households earn labor income, which is subject to an aggregate wage shock. Out of this labor income, they can consume and save. There is a single asset whose return is stochastic, which represents a second aggregate risk. The second period of life consists of two subperiods. In the first, households again learn labor income, but now receive an idiosyncratic productivity shock in addition to the

[1] See, e.g., Krueger and Kubler (2006), Ludwig and Reiter (2010), Hasanhodzic and Kotlikoff (2013) for social security analyses in settings with aggregate risk where social security improves intergenerational risk sharing and İmrohoroğlu, İmrohoroğlu, and Joines (1995, 1998) and Conesa and Krueger (1999) as examples of studies where social security provides insurance against idiosyncratic productivity risk.

aggregate wage shock. In the second subperiod, households retire and receive pension income. We construct the model in general equilibrium by assuming a representative firm with a standard neo-classical production function. Production is subject to aggregate business cycle risk which gives rise to the aggregate fluctuations of wages and asset returns.[2] A crucial assumption maintained throughout is that all shocks are mutually orthogonal, i.e., they are statistically independent of each other so that there is no direct interaction between the risks. Social security in our model is a pure pay-as-you-go (PAYG) system with defined contributions and a lump-sum pension. With this design, the system partially insures both aggregate as well as idiosyncratic risks. Our thought experiment considers the introduction of a marginal social security system of this design. Hence, we study the welfare implications of a flat minimum pension which we evaluate using an ex-ante Utilitarian welfare criterion.[3]

Our first set of results looks at insurance provided through social security and how it is affected when two risks are present. To this end we study a partial equilibrium version of the model, and we assume that households only consume in the second period of life.[4] As our first main finding we establish that joint presence of both risks increases the welfare benefits from insurance against old-age consumption risk more than additively. Hence the whole welfare benefit from insurance is greater than the sum of welfare gains from insurance against isolated risk components. We also speak of this welfare difference between the whole effect and the sum of its parts as resulting from (positive) *risk interactions*, bearing in mind that risk interactions are indirect here in that they operate through the utility function or the social welfare function.[5]

Our second set of results characterizes how these risk interactions affect the welfare costs of crowding out of capital formation. A higher contribution rate distorts the savings decision and therefore leads to crowding out of aggregate capital. Since we assume that the economy is dynamically efficient, the crowding out leads to welfare losses. Our central

[2]This general equilibrium model can be seen as an extension of the standard Diamond (1965) model with aggregate and idiosyncratic risk. The setup is similar to Huffman (1987) with three important differences: First, we extend his work by taking into account idiosyncratic risk. Second, we don't only consider positive labor income in the first period of life but rather have two periods with positive labor income. Third, we stick to a two period structure while Huffman (1987) has many periods.

[3]Most real world pension systems feature some distributional components. Almost all systems have a minimum pension. In fact, our system features strong similarities to the Danish public pension system.

[4]This assumption is also made by Gordon and Varian (1988), Ball and Mankiw (2007), Matsen and Thogersen (2004), Krueger and Kubler (2006), Harenberg and Ludwig (2015), among others.

[5]This terminology is borrowed from statistical data analysis. To measure how both risks increase welfare gains and whether there is more than an additive effect, an econometrician would consider in a (linear) regression as an *interaction term* the product of risk measures, i.e., the product of variances.

result here is that when idiosyncratic risk is increased, the welfare losses from crowding out are determined by two opposing forces. On the one hand, increasing idiosyncratic risk leads to larger crowding out, because the marginal introduction of social security now (partially) insures a larger risk. This increases the welfare losses from crowding out. On the other hand, higher idiosyncratic productivity risk increases precautionary savings, so that households will profit more from the higher interest rate that results from the crowding out of aggregate capital. An additional key result is that the interactions enter both of these forces so that it is ambiguous whether they amplify or mitigate the losses from crowding out. Thus, while the insurance gains of social security are unambiguously increased through the interactions, the result for the welfare losses is ambiguous. We therefore conclude that it is a quantitative question whether social security can ultimately increase welfare in economies with both risks.

The present paper is closely related to our quantitative work in Harenberg and Ludwig (2015), abbreviated by HL in the following. HL show that appropriately taking into account both risks indeed substantially alters the quantitative welfare implications of a social security system which partially insures both risks. They document welfare losses in economies with only a single risk, while reporting strong welfare gains when both risks are simultaneously at work. Importantly, they assign a large welfare enhancing role to the indirect risk interactions. To motivate the quantitative analysis, HL also consider a simple two-period example in partial equilibrium and derive closed form solutions for the insurance benefits from social security. Relative to that, the theoretical model developed here features three main differences. First, we generalize the setup of HL by allowing households to consume in both periods. As a consequence, the endogenous savings choice becomes non-trivial. Second, and most importantly, we develop a general equilibrium model which enables us to characterize the net welfare effects of social security. Third, HL also consider direct interactions between the risks.[6]

Our work is closely related to the theoretical background risk literature, see, e.g., Gollier and Pratt (1996). That literature started by considering decision situations where households choose exposure to a market risk, when an additive mean zero background risk is added. Franke, Schlesinger, and Stapleton (2006) extend this by a multiplicative background risk, which means that the market risk is multiplied with an independent risk, and Franke, Schlesinger, and Stapleton (2011) combine both setups by considering

[6]This is done in HL by conditioning higher moments of idiosyncratic productivity risk on the aggregate state of the economy via a countercyclical cross-sectional variance of idiosyncratic productivity risk, cf., e.g., Mankiw (1986), Constantinides and Duffie (1996) ,Storesletten, Telmer, and Yaron (2004), Storesletten, Telmer, and Yaron (2007) and, more recently, Guvenen, Ozkan, and Song (2014).

additive and multiplicative background risk. This literature is concerned with static decision situations whereas we look at a dynamic model. We show that a situation with additive and multiplicative background risk naturally arises in such a dynamic decision model when households have both risky wage and asset income and when wage income features an idiosyncratic and an aggregate risk component. Two additional important differences stand out. First, in our setup, a social planner chooses to implement social security. Hence, the implicit portfolio choice—the fraction of implicit savings in social security—is not made by the household. Second, as the most important difference to that work, social security reduces exposure to both the market and the background risk jointly whereas in the background risk literature only the exposure to the market risk can be reduced.

Our work also relates to the literature on the welfare costs of aggregate fluctuations initiated by Lucas (1978). De Santis (2007) and Krebs (2007) demonstrate that interactions between idiosyncratic and aggregate risk can increase these costs substantially. Relative to this work, again the key difference here is that we study the effects of joint insurance against both risks. Finally, our work relates to a large theoretical and quantitative literature on the welfare benefits of social security which we discuss in depth in Harenberg and Ludwig (2015).

Our analysis proceeds as follows. Section 2 presents the model which is analyzed in Section 3. Section 4 provides a numerical illustration. Section 5 concludes. All proofs are relegated to Appendix A. Supplementary Appendix B—available on our web-pages—contains additional results.

2 The Model

2.1 Time and Population

Time is discrete. Periods in our model are denoted by $t = -\infty, \ldots, 0, 1, \ldots, \infty$. In each period, two generations—the young, indexed by $j = 1$, and the old, indexed by $j = 2$—are simultaneously alive. Each generation consists of a continuum of households. We consider a stationary population.

In our setup, a PAYG pension system would not provide insurance against the risk of longevity even when annuity markets are missing as long as accidental bequests are redistributed, as was shown by Caliendo, Guo, and Hosseini (2014). We therefore do not model survival risk which would, in any case, lead us on a sidetrack. Denoting the period t

young population by $N_{1,t}$ and the old by $N_{2,t}$ we accordingly have that $N_{2,t} = N_{1,t}$.

As there is idiosyncratic risk to labor income, we further distinguish by types. We assume that the idiosyncratic productivity shocks hit households only in the second period of their lives. In our partial equilibrium analysis, this assumption allows us to study background risk in a tractable and clear way. In the general equilibrium it is necessary to characterize the dynamics in closed form.[7,8] As households are ex-ante identical, the type distinction is only needed for the second period. We denote by $N_{i,2,t}$ the number of households of type i of age 2 alive in period t and have $N_{t,2} = \int N_{i,2,t} di$. We normalize the population of age j to unity, hence $N_{t,j} = 1$ for $j = 1, 2$.

2.2 Households

A household has preferences over consumption in two periods. The expected utility function of a household in period t is given by

$$U_t = (1 - \tilde{\beta}) u(c_{1,t}) + \tilde{\beta} \mathbb{E}_t \left[u(c_{2,t+1}) \right],$$

where the per period Bernoulli utility function u is (weakly) increasing and concave, i.e., $u' \geq 0, u'' < 0$. Expectations in the above are taken with respect to the idiosyncratic productivity shock as well as aggregate wage and return shocks to be specified below. In our notation, we make explicit that households form expectations conditional on the information at their date of birth and therefore denote the expectations operator $\mathbb{E}$ with subscript t. As these expectations are formed at the beginning of period t, realizations of shocks in period t are in the information set. The factor $\tilde{\beta} \leq 1$ determines the relative weight on first versus second period (expected) utility from consumption, and for $\tilde{\beta} \neq 1$, $\beta \equiv \frac{\tilde{\beta}}{1-\tilde{\beta}}$ is the discount factor.

We assume that the per period utility function u is CRRA with coefficient of relative

[7] In our proof of equilibrium dynamics, we require a homothetic structure. We do not get that with idiosyncratic risk in the first period and a lump-sum pension payment in the second, because the first-period wage poor save less than the first-period wage rich. This could be made homothetic by assuming that pension payments do not redistribute across types but then social security no longer insures against idiosyncratic risk.

[8] For results when households face idiosyncratic risk only in the first period of life, see Harenberg and Ludwig (2015).

risk aversion θ:

$$u(c_{i,j,t}) = \begin{cases} \frac{c_{i,j,t}^{1-\theta}}{1-\theta} & \text{for } \theta \neq 1 \\ \ln(c_{i,j,t}) & \text{for } \theta = 1, \end{cases} \tag{1}$$

where it is understood that the type index i is only relevant for $j = 2$.

Households work full time in the first period. For the second period of life, we follow Auerbach and Hassett (2007), Ludwig and Vogel (2010), and others and consider a subperiod structure. In the first subperiod—which is of relative length $\lambda \in [0, 1)$—households work. We also refer to λ as labor productivity in the second period. In the second subperiod—of length $1 - \lambda$—households are retired and receive a pension income $b_t \geq 0$. The subperiod structure is convenient for analytical reasons. Combined with idiosyncratic income shocks in the second period it enables us to model precautionary savings together with retirement savings without having to introduce a three-generations structure. This preserves simple first-order difference equations in our characterization of equilibrium dynamics of the economy.[9] The budget constraints in the two periods are accordingly given by

$$c_{1,t} + a_{2,t+1} = (1 - \tau)w_t \tag{2a}$$

$$c_{i,2,t+1} \leq a_{2,t+1}(1 + r_{t+1}) + \lambda\eta_{i,2,t+1}w_{t+1}(1 - \tau) + (1 - \lambda)b_{t+1}\,, \tag{2b}$$

where $\eta_{i,2,t}$ is the age-2, period-t idiosyncratic shock to wages, and $a_{2,t+1}$ denotes savings of a young household, which equal his asset position at the beginning of the following period. Finally, τ is the (constant) social security contribution rate.

2.3 Government

The government organizes a PAYG financed social security system. Pension benefits are lump-sum. Therefore, idiosyncratic wage risk is insured through social security. Each period the mass of workers who earn aggregate gross wages w_t is $L = 1 + \lambda$. The mass of pensioners is $1 - \lambda$. The social security budget constraint is then $b_t = \tau w_t \frac{1+\lambda}{1-\lambda}$.

[9] In the Supplementary Appendix B.1, we show that the results from Subsection 3.1 would go through in a three-generations model.

2.4 Firms

To close the model in general equilibrium, we add a firm sector. We assume a rental market with a static optimization problem. Firms maximize profits operating a neo-classical production function. Let profits of the firm be

$$\Pi = \zeta_t F(K_t, \Upsilon_t L) - (\bar{\delta} + r_t)\varrho_t^{-1} K_t - w_t L,$$

where ζ_t is a technology shock, K_t is the beginning of period t capital stock, and L is total labor which equals $L = 1 + \lambda$. The technology level, Υ_t, grows at an exogenous rate g, $\Upsilon_t = (1+g)\Upsilon_{t-1}$, for a given Υ_0. Throughout we assume full depreciation, hence $\bar{\delta} = 1$. The variable ϱ_t represents an exogenous shock to the unit user costs of capital. We add this non-standard element in order to model additional shocks to the rate of return to capital. These shocks are multiplicative in the user costs of capital for analytical reasons. Production is Cobb-Douglas with capital elasticity α, $F(K_t, \Upsilon_t L) = K_t^{\alpha} (\Upsilon_t L)^{1-\alpha}$. Let $k_t = \frac{K_t}{\Upsilon_t L}$ be the capital intensity, i.e., the capital stock per efficient unit of labor. Then, the firm's first-order conditions are

$$R_t = 1 + r_t = \alpha k_t^{\alpha-1} \zeta_t \varrho_t = \bar{R}_t \zeta_t \varrho_t \tag{3a}$$

$$w_t = (1-\alpha)\Upsilon_t k_t^{\alpha} \zeta_t = \bar{w}_t \zeta_t, \tag{3b}$$

where $\bar{R}_t$ denotes the non-stochastic component of the gross return and, likewise, $\bar{w}_t$ the non-stochastic component of the per capita wage. Equation (3a) reveals that ϱ_t is simply a shock to the gross return on savings, since it does not affect wages.

2.5 Social Welfare and Thought Experiment

We take an ex-ante Rawlsian perspective and specify the social welfare function as

$$SWF \equiv \mathbb{E}U_t = \mathbb{E}\left[(1-\tilde{\beta})u(c_{1,t}) + \tilde{\beta}u(c_{2,t+1})\right]. \tag{4}$$

We consider a marginal introduction of social security and investigate how SWF is affected by such a policy reform. More precisely, we compare social welfare and the sources of welfare gains and losses in two stationary equilibria, one without social security and one with a marginal social security system. By ignoring transitional dynamics we exaggerate the welfare losses of crowding out experienced by generations born during the transition. This is so because the gains from insurance of a reform materialize on impact whereas

the complete losses from crowding out only occur in the limit when the new steady state is reached.

2.6 Stochastic Processes

To simplify the analysis we assume that both ζ_t and ϱ_t are not serially correlated. Despite the observed positive serial correlation of wages and asset returns in annual data, this assumption can be justified on the grounds of the long factual periodicity of each period in a two-period OLG model which is about 30 to 40 years. We also assume that ζ_t and ϱ_t are statistically independent so that dependence of return and wage shocks is only reflected through ζ_t. The idiosyncratic shock $\eta_{i,2,t}$ is not correlated with either of the two aggregate shocks.

Assumption 1. *a) Support bounded from below:* $\zeta_t > 0$, $\varrho_t > 0$, *and* $\eta_{i,2,t} > 0$ *for all* i, t.

b) Means: $\mathbb{E}\zeta_t = \mathbb{E}\varrho_t = \mathbb{E}\eta_{i,2,t} = 1$, *for all* i, t.

c) Statistical independence of (ζ_{t+1}, ζ_t) *and* $(\varrho_{t+1}, \varrho_t)$. *Therefore:* $\mathbb{E}(\zeta_{t+1}\zeta_t) = \mathbb{E}\zeta_{t+1}\mathbb{E}\zeta_t$ *for all* t *and, correspondingly,* $\mathbb{E}(\varrho_{t+1}\varrho_t) = \mathbb{E}\varrho_{t+1}\mathbb{E}\varrho_t$ *for all* t.

d) Statistical independence of (ζ_t, ϱ_t). *Therefore:* $\mathbb{E}(\zeta_t\varrho_t) = \mathbb{E}\zeta_t\mathbb{E}\varrho_t$ *for all* t.

e) Statistical independence of $(\zeta_t, \eta_{i,2,t})$. *Therefore:* $\mathbb{E}(\eta_{i,2,t}\zeta_t) = \mathbb{E}\eta_{i,2,t}\mathbb{E}\zeta_t$ *for all* i, t.

f) Statistical independence of $(\varrho_t, \eta_{i,2,t})$. *Therefore:* $\mathbb{E}(\eta_{i,2,t}\varrho_t) = \mathbb{E}\eta_{i,2,t}\mathbb{E}\varrho_t$ *for all* i, t.

3 Analysis

In this section, we first analyze a partial, then a general equilibrium.

3.1 Partial Equilibrium

In a partial equilibrium, wages and returns are completely exogenous. This allows us to specify directly the stochastic processes driving them. In particular, it allows us to model wages and returns as uncorrelated. Whereas in general equilibrium, both shocks ζ_t and ϱ_t affect returns, we here assume that there is a separate return shock $\tilde{\varrho}_t$ which is independent of ζ_t. In addition, we assume that households only care about second period consumption. This helps to focus on the insurance that social security provides, but will

change in the general equilibrium section, where we allow a consumption-savings choice in the first period. The following assumption summarizes this.

Assumption 2. *a) Let $k_t = \bar{k}$ given, hence $w_t = \bar{w}_t\zeta_t = \bar{w}_{t-1}(1+g)\zeta_t$ and $R_t = \bar{R}\tilde{\varrho}_t$, where $\tilde{\varrho}_t$ has the stochastic properties of ϱ_t from Assumption 1.*

b) Let $\tilde{\beta} = 1$.

We can now rewrite consumption in the second period of a household's life as

$$c_{i,2,t+1} = \bar{w}_t \Big(\zeta_t \bar{R}\tilde{\varrho}_{t+1} + (1+g)\zeta_{t+1}\eta_{i,2,t+1}\lambda + \tau\left((1+g)\zeta_{t+1}\left(1+\lambda(1-\eta_{i,2,t+1})\right) - \zeta_t\bar{R}\varrho_{t+1}\right)\Big). \tag{5}$$

Let's start by looking at a situation where $\tau = 0$ and, without loss of generality, $\bar{w}_t = 1$. Then, old age consumption becomes

$$c_{i,2,t+1} = \underbrace{\zeta_t\tilde{\varrho}_{t+1}}_{\equiv\phi}\bar{R} + (1+g)\lambda + (1+g)\lambda(\underbrace{\zeta_{t+1}\eta_{i,2,t+1}}_{\equiv\psi} - 1). \tag{6}$$

This is formally equivalent to a situation with additive and multiplicative background risk, similar to Franke, Schlesinger, and Stapleton (2011).[10] A major difference is that they look at a static model where the background risk is additive and multiplicative by construction. By contrast, in our dynamic setting the multiplicative background risk arises endogenously due to the dynamic structure of the economy.

For our purpose, note that both ψ and ϕ implicitly have an interaction term[11], which can be seen by expanding their variances, $\text{var}(\psi) = \text{var}(\zeta_{t+1}\eta_{i,2,t+1}) = \sigma_\zeta^2 + \sigma_\eta^2 + \sigma_\zeta^2\sigma_\eta^2$ and $\text{var}(\phi) = \text{var}(\zeta_t\tilde{\varrho}_{t+1}) = \sigma_\zeta^2 + \sigma_\varrho^2 + \sigma_\zeta^2\sigma_\varrho^2$.[12] Of course, such an interaction would not be present if we measured the variances in logs of the respective random variables rather than in levels. However, as we formally show in Harenberg and Ludwig (2015), the product of variances in levels is a convenient way to express how the joint presence of both risks over-proportionally increases the risk exposure of households. This makes utility losses from the exposure to risk increase more than additively in both risks. Thus, the interaction terms capture how the value of a marginal introduction of social security

[10] The independent, mean-zero shock ζ_t is the multiplicative background risk, because it multiplies the market risk $\tilde{\varrho}_{t+1}\bar{R}$. The independent, mean-zero shock $\zeta_{t+1}\eta_{i,2,t+1}$ is the additive background risk.

[11] We label the product of any two variables x_1 and x_2 an "interaction term" as it is common in statistical data analysis.

[12] See the product formula of variances derived in Goodman (1960).

increases from the ex-ante perspective, because they increase the variance of retirement consumption.

We can shut down the interaction in both ψ and ϕ by assuming $\sigma_\zeta = 0$, i.e., $\zeta_t = E\zeta_t = 1$ for all t. Coincidentally, we then have the more well-known situation with only additive background risk that was originally considered by Gollier and Pratt (1996).

The main difference to this background risk literature is the thought experiment. The background risk literature asks how behavior with respect to a market risk changes when a zero mean background risk is added. A general finding is that households will then behave more risk averse. The mirror image of this is that they value insurance against the market risk more when the background risk is added. Our setup differs because we ask how the valuation of insurance is affected by both risks when both risks are simultaneously insured. With respect to this thought experiment, we get for a marginal introduction of social security the following result:

Proposition 1. *Under Assumptions 1 and 2, a marginal introduction of social security increases social welfare iff*

$$A_{pe} = \mathbb{E}\left[\frac{\frac{1+g}{\bar{R}}(1+\lambda)\frac{\zeta_{t+1}}{\zeta_t\varrho_{t+1}} - \frac{1+g}{\bar{R}}\lambda\frac{\zeta_{t+1}\eta_{i,2,t+1}}{\zeta_t\varrho_{t+1}} - 1}{\left(\frac{1}{\bar{R}\zeta_t\varrho_{t+1}}\right)^{1-\theta}\left(1+\frac{1+g}{\bar{R}}\lambda\frac{\zeta_{t+1}\eta_{i,2,t+1}}{\zeta_t\varrho_{t+1}}\right)^{\theta}}\right] > 0 \tag{7}$$

Proof. See Appendix A. □

In order to simplify the following analysis, we concentrate on the case where $\theta = 1$ to the effect that equation (7) becomes

$$A_{pe}|_{\theta=1} = \mathbb{E}\left[\frac{\frac{1+g}{\bar{R}}(1+\lambda)\frac{\zeta_{t+1}}{\zeta_t\varrho_{t+1}} - \frac{1+g}{\bar{R}}\lambda\frac{\zeta_{t+1}\eta_{i,2,t+1}}{\zeta_t\varrho_{t+1}} - 1}{1+\frac{1+g}{\bar{R}}\lambda\frac{\zeta_{t+1}\eta_{i,2,t+1}}{\zeta_t\varrho_{t+1}}}\right] > 0. \tag{8}$$

Observe that term $\frac{1+g}{\bar{R}}$ in equation (8) reflects the well-known trade-off between an implicit investment in social security and an explicit investment in a risk-free asset. It is the standard Aaron condition (Aaron 1966), which in our context says that in a risk-free environment, an introduction of social security is welfare increasing if and only if $\frac{1+g}{\bar{R}} > 1$. The other terms in equation (8) represent a risk adjustment which scales up the implicit return of social security, $1+g$. The proposition states that if there is sufficient risk, then the introduction of social security may improve welfare even when the deterministic version of the economy has $\bar{R} > 1+g$. As we generally assume that $\theta \geq 1$, this constitutes a lower bound for A_{pe} because welfare benefits increase in θ.

To investigate how interactions of risks affect the term A_{pe}, we analyze its cross-derivatives. To derive expressions in closed form we now need to assume log-normality and consider a Taylor-series approximation.

Assumption 3. *Log-normality:* $\eta_{i,1,t}, \zeta_t, \zeta_{t+1}, \tilde{\varrho}_{t+1}$ *are distributed as log-normal with parameters* $\mu_{\ln\eta}$, $\mu_{\ln\zeta}$, $\mu_{\ln\tilde{\varrho}}$, $\sigma^2_{\ln(\eta)}$, $\sigma^2_{\ln(\zeta)}$, $\sigma^2_{\ln(\tilde{\varrho})}$ *for means and variances, respectively.*

Proposition 2. *Consider* $\theta = 1$. *Under Assumptions 1, 2, and 3, a second-order Taylor series expansion yields* $\frac{\partial A_{pe}|_{\theta=1}}{\partial\sigma^2_\eta}$, $\frac{\partial^2 A_{pe}|_{\theta=1}}{\partial\sigma^2_\eta\partial\sigma^2_\zeta} > 0$, *and* $\frac{\partial^2 A_{pe}|_{\theta=1}}{\partial\sigma^2_\eta\partial\sigma^2_{\tilde{\varrho}}} > 0$.

Proof. See Appendix A. □

Thus, as idiosyncratic risk increases, the insurance benefits of social security go up. More interestingly, this slope is larger, the larger the aggregate risk. This is so because from an ex-ante perspective, all the risks increase the variance of retirement consumption. Since the utility function is concave and has the Inada-properties, an increase in the variance of consumption translates into larger utility losses. Social security is beneficial because it (partially) insures against both risks. The fact that the two types of risk amplify the welfare benefits of such a social security system is a central result of this paper.

3.2 General Equilibrium

The previous analysis is restricted to the special case with zero consumption in the first period. In that setting, the value of social security stems from insurance against the risk of income fluctuations. The costs stem from the fact that in a dynamically efficient economy, gross market returns are higher than the implicit return of a PAYG social security system. An important channel is missing in that setting. To the extent that social security reduces consumption risk, households need to save less for precautionary motives and they also save less for life-cycle motives. By crowding out savings, the expansion of social security reduces the aggregate capital stock which suppresses wages and increases returns. This reduces welfare in a dynamically efficient economy. As we will see, as the second central result of this paper, the interactions of risks can amplify or mitigate the welfare costs of crowding out.

In order to illustrate this additional channel, we consider a setting where consumption decisions are also made in the first period and embed the analysis into a general equilib-

rium model. For analytical reasons we have to incorporate both steps at once.[13] We also have to restrict attention to log-utility[14] in both periods. This is summarized in the next assumption, which replaces Assumption 2 of the previous partial equilibrium section.

Assumption 4. *a)* $u(\cdot) = \ln(\cdot)$

b) $\tilde{\beta} \in \left(0, \frac{1}{2}\right] \Leftrightarrow \beta = \frac{\tilde{\beta}}{1-\tilde{\beta}} \in (0, 1]$

General Equilibrium Dynamics

We begin by characterizing the equilibrium dynamics of the economy.

Proposition 3. *Under Assumptions 1, and 4, equilibrium dynamics are given by*

$$k_{t+1} = \frac{1}{(1+g)(1+\lambda)} s(\tau)(1-\tau)(1-\alpha)\zeta_t k_t^{\alpha} \tag{9}$$

for some initial capital stock k_0*. The saving rate is given by*

$$s(\tau) \equiv \frac{\beta\Phi(\tau)}{1+\beta\Phi(\tau)} \leq \frac{\beta}{1+\beta}, \tag{10}$$

where

$$\Phi(\tau) \equiv \mathbb{E}_t \left[\frac{1}{1 + \frac{1-\alpha}{\alpha(1+\lambda)\varrho_{t+1}} \left(\lambda\eta_{i,2,t+1} + \tau\left(1 + \lambda(1-\eta_{i,2,t+1})\right)\right)} \right] \leq 1. \tag{11}$$

Proof. See Appendix A. □

Notice from (10) that an increase of Φ increases the saving rate. Turning to equation (11), first consider a risk-free situation ($\eta_{i,2,t+1} = \varrho_{t+1} = 1$) without a pension system ($\tau = 0$). We then have $\Phi = \frac{1}{1+\frac{1-\alpha}{\alpha}\frac{1}{1+\frac{1}{\lambda}}}$. An increase of λ leads to higher wage income in the second period (and a shorter retirement subperiod) which decreases the saving rate by decreasing Φ . The textbook model with log-utility and Cobb-Douglas production is nested for $\lambda = 0$ where $\Phi = 1$ and the saving rate is constant at $\frac{\beta}{1+\beta}$.

[13] In a partial equilibrium model with pension income in the second period—and/or with positive second period labor income in case $\lambda > 0$—, the human capital wealth effect inhibits closed form solutions for the saving rate. Our proof of equilibrium dynamics uses the fact that both the interest rate and the wage rate, on which pension payments are based, are functions of the capital stock in general equilibrium. This enables us to conveniently rewrite the discounted value of second period labor income (=human capital) so that we can derive closed form solutions for the saving rate and the equilibrium dynamics.

[14] It is crucial that income and substitution effects of changing interest rates offset each other.

Next, let's look at $\lambda > 0$ and introduce risk while keeping $\tau = 0$. Then $\Phi = \mathbb{E}_t\left[\frac{1}{1+\frac{1-\alpha}{\alpha}\frac{\lambda}{1+\lambda}\frac{\eta_{i,2,t+1}}{\varrho_{t+1}}}\right]$. Now a mean preserving spread of idiosyncratic shocks, $\eta_{i,2,t+1}$, increases Φ thereby increasing the saving rate, s, as long as $\lambda > 0$. This is precautionary savings. By contrast, an increase in the variance of return shocks, ϱ_t, reduces Φ thereby decreasing the saving rate, s. The reason is simply that the asset becomes less attractive, since its risk goes up while the return remains the same.

Finally, let's consider $\tau > 0$. Increasing τ decreases Φ and therefore decreases the saving rate, s. This is the crowding-out of private capital formation. Moreover, the larger τ, the smaller the effect of a mean preserving spread of $\eta_{i,2,t+1}$ on precautionary savings, because of the insurance provided through social security. In the limit case where $\tau = 1$, $\eta_{i,2,t+1}$ has no effect on the saving rate.

Welfare Analysis

We now turn to a central section of the paper, the welfare analysis in general equilibrium. We look at the same experiment as before, a marginal introduction of a PAYG social security system. In general equilibrium, we can oppose the welfare gains from insurance that we analyzed in the previous section with the potential welfare losses due to the crowding out of capital.

Proposition 4. *Under Assumptions 1 and 4, a marginal introduction of social security increases social welfare in the stationary equilibrium iff*

$$A + B > 0$$

where

$$A \equiv \beta\mathbb{E}\left[\frac{\frac{(1-\alpha)}{\alpha}\frac{1}{\varrho_{t+1}} - \frac{(1-\alpha)\lambda}{\alpha(1+\lambda)}\frac{\eta_{i,2,t+1}}{\varrho_{t+1}} - 1}{1 + \frac{(1-\alpha)\lambda}{\alpha(1+\lambda)}\frac{\eta_{i,2,t+1}}{\varrho_{t+1}}}\right] - 1 \tag{12}$$

$$B \equiv -\frac{1}{1-\alpha}\left(1 - \epsilon_{s,\tau}|_{\tau=0}\right)\left(\alpha(1+\beta) - \beta(1-\alpha)\,\Phi|_{\tau=0}\right) \tag{13}$$

where Φ is shown in equation (11) and $\epsilon_{s,\tau} \equiv \frac{\partial s/s}{\partial \tau} < 0$ is the semi-elasticity of the saving rate with respect to the contribution rate.

Proof. See Appendix A. □

In the above, term A reflects the rate of return condition of social security and thus is

the general equilibrium analogue to the partial equilibrium term A_{pe} from equation (8). There are two differences between the two. First, in general equilibrium, the interest rate is determined endogenously. Second, in general equilibrium the aggregate productivity shocks, ζ_t and ζ_{t+1}, drop out. Intuitively, this happens because that shock affects all sources of income, namely wages, returns, and social security pensions.

Analogous to the analysis of the partial equilibrium analysis, term A depicts the trade-off between the insurance gains due to social security and the welfare losses due to the fact that the implicit return of social security is less than the expected return on savings in a dynamically efficient economy. We make two more observations regarding term A. First, term A does not capture any behavioral responses to the policy reform. Second, it increases in β, because households care more about consumption risk in the second period when β is higher.

Term B represents the welfare effects due to crowding out of capital formation. It thus captures the response of households to the introduction of the pension system. Whether term B is positive or negative depends on whether the economy is dynamically efficient. This is formalized in the next proposition.

Lemma 1. *Consider a deterministic economy with $\lambda = 0$. This economy is dynamically efficient in the sense of Cass (1972) iff*

$$s(\tau = 0, \lambda = 0) = \frac{\beta}{1+\beta} < \frac{\alpha}{1-\alpha}. \tag{14}$$

Proposition 5. *If condition (14) holds in the deterministic economy with $\lambda = 0$, then term $B < 0$ in the corresponding stochastic economy with $0 \leq \lambda < 1$.*

Proof. See Appendix A. The lemma is proved as part of the proposition. □

The proposition connects the classic notion of dynamic efficiency due to Cass (1972) to the welfare effect of crowding out in our stochastic economy. If the deterministic version of the economy is dynamically efficient, then the crowding out of capital leads to a welfare loss, i.e., term $B < 0$.

We are now in a position to discuss in detail term B in equation (13). The first part $-\frac{1}{1-\alpha}\left(1 - \epsilon_{s,\tau}|_{\tau=0}\right)$ captures the effects of social security on the allocation, i.e., the crowding out of the capital stock. Crowding out is caused by lower first period wage income because of social security taxation (its accumulated effect is reflected by $\frac{1}{1-\alpha}$) and the decrease of the saving rate in response to the introduction of social security (reflected by $1 - \epsilon_{s,\tau}|_{\tau=0}$, with $\epsilon_{s,\tau}|_{\tau=0} < 0$). The second part, $\alpha(1+\beta) - \beta(1-\alpha)\,\Phi|_{\tau=0}$, captures

how this change in the allocation translates into welfare consequences. There are two effects at work. To understand those, first consider the textbook model with $\lambda = 0$ so that $\Phi|_{\tau=0} = 1$. Then $\alpha(1+\beta)$ captures the negative effects on welfare of a crowding out of capital because the entire consumption path is shifted down (the wage effect), while term $-\beta(1-\alpha)$ captures how the crowding out of capital tilts the consumption profile by increasing the interest rate (the interest rate effect). In a dynamically efficient economy, the wage effect dominates because $\alpha(1+\beta) - \beta(1-\alpha) > 0$.

From condition (14) it is also readily observed that the welfare costs of crowding out ceteris paribus decrease in β. For more patient (high β) households a reduction of the capital stock is less painful because of the partial compensation through the positive interest rate effect. It is worth emphasizing the symmetry: increasing β means that welfare benefits from insurance are valued more (term A) and costs from crowding out decrease (term B).

Next turn to the $\lambda > 0$ economy, assuming for now that the economy is deterministic ($\sigma_\eta^2 = 0, \sigma_\varrho^2 = 0$). As households have positive wage income in the second period with relative importance governed by $\lambda > 0$, the negative utility consequences of a decreasing capital stock are increased relative to the $\lambda = 0$ case. That is, the strength of the interest rate effect decreases relative to the wage effect. Formally, term $\Phi|_{\tau=0}$ captures this as it decreases in λ. A decrease of $\Phi|_{\tau=0}$ means that households save less, cf. our discussion of Proposition 3, hence the interest rate effect looses importance relative to the wage effect.

To sum up the discussion on the two terms A and B for now, in a dynamically efficient economy, the introduction of social security may increase welfare due to insurance, reflected by term A, but it reduces welfare due to the crowding out of capital, reflected by term B. In the following, we discuss how risk and the interactions of risks affect the two terms. To this end, we analyze the derivatives $\frac{\partial A}{\partial \sigma_\eta^2}$ and $\frac{\partial B}{\partial \sigma_\eta^2}$ as well as the cross partial derivatives, $\frac{\partial^2 A}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2}$ and $\frac{\partial^2 B}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2}$. If the latter are positive, then idiosyncratic and aggregate risks interact positively, just as in our previous partial equilibrium analysis of Subsection 3.1. We again consider Taylor series expansions of the respective random variables around their respective means. We need to modify Assumption 3 on the log-normality of shocks to take into account the random variable ϱ which replaced $\tilde{\varrho}$.[15]

Assumption 5. *Log-normality: $\eta_{i,2,t}, \varrho_{t+1}$ are distributed as log-normal with parameters $\mu_{\ln \eta}$, $\mu_{\ln \varrho}$, $\sigma^2_{\ln(\eta)}$, $\sigma^2_{\ln(\varrho)}$ for means and variances, respectively.*

[15]In the Supplementary Appendix B.2, we examine the special case of $\lambda = 0$, which yields concise equations without the need for an additional assumption.

Proposition 6. *Under Assumptions 1, 4, and 5, a second-order Taylor series expansion yields* $\frac{\partial A}{\partial \sigma_\eta^2} > 0$, $\frac{\partial A}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} > 0$, $\frac{\partial \Phi|_{\tau=0}}{\partial \sigma_\eta^2} > 0$, $\frac{\partial^2 \Phi|_{\tau=0}}{\partial \sigma_\varrho^2 \partial \sigma_\eta^2} > 0$, $\frac{\partial B}{\partial \sigma_\eta^2} \gtreqless 0$, *and* $\frac{\partial B}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} \gtreqless 0$.

Proof. See Appendix A. □

Our results with regard to term A are analogous to our partial equilibrium results from Subsection 3.1. Welfare benefits from introducing social security are increasing in the amount of idiosyncratic risk, and the slope interacts positively with aggregate risk.

In contrast, the effect of idiosyncratic risk on the welfare losses from crowding out, term B, is ambiguous. First, we cannot (in general) sign $\frac{\partial \epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2}$. We will focus on the economically relevant case where the derivative is negative. Intuitively, household saving reacts more strongly to the introduction of social security when idiosyncratic risk is higher because then precautionary savings are reduced more strongly in response to better consumption insurance. In fact, we always found this to be the case throughout all numerical exercises we performed.[16] Hence, the crowding out of capital is stronger when idiosyncratic risk increases. This, ceteris paribus, causes a reduction of utility.

Second, there is an important opposing effect at work. Increasing idiosyncratic risk means that households save more out of precautionary motives. This means that the strength of the interest rate effect of crowding out—formally captured by term $\beta(1-\alpha)\,\Phi|_{\tau=0}$ in equation (13)—increases relative to the wage effect (term $\alpha(1+\beta)$ in equation (13)). Formally this is reflected by $\frac{\partial \Phi|_{\tau=0}}{\partial \sigma_\eta^2} > 0$, cf. our discussion of Proposition 4. Therefore, while an increase of idiosyncratic risk leads to a stronger reduction of the capital stock in response to the introduction of social security, a given reduction translates less into welfare so that the net welfare effect of the increased crowding out is ambiguous.

With respect to the cross-derivative this ambiguity continues to hold. First, similar to before, we assume that $\frac{\partial^2 \epsilon_{s,\tau}}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} < 0$. Again we cannot show this analytically but this is confirmed in all our numerical exercises and captures the notion that precautionary savings are reduced more strongly when both risks are jointly insured. Second, the ambiguity in $\frac{\partial B}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2}$ is now due to the following effect. In case of an increase of return risk, households increase their precautionary savings more strongly in response to an increase of idiosyncratic wage risk than without that additional return risk. Hence the interest rate effect becomes more important. Formally, this is captured by the positive cross partial of $\Phi|_{\tau=0}$ which increases the saving rate. Observe the analogy to the standard intuition from the additive background risk literature, cf. Gollier and Pratt (1996): in

[16] In our proof of Proposition 6 we also characterize a lower bound on the semi-elasticity such that the derivative is negative.

that context, households behave more risk averse (and value insurance more) when an additive background risk (which here is the return risk) is increased. In the current context, households save more for precautionary motives.

It is worth emphasizing the asymmetry in the welfare benefits and welfare costs of social security. As idiosyncratic risk is increased, the benefits unambiguously increase, whereas the costs may in- or decrease. To the very least, there is an important dampening mechanism. The increase in benefits will be larger the larger the level of aggregate risk, but again this is not clear for the costs. It suggests that the case for social security might be stronger in an economy in which both risks are modeled.[17] However, also recall from our discussion of Proposition 4 that the relative size of the benefits from insurance and the welfare losses from crowding out crucially depend on discounting. We address these aspects in our numerical analysis that follows next.

4 Numerical Illustration

This section provides a numerical illustration of the general equilibrium results presented in Proposition 4. The aim is not to perform a rigorous quantitative exercise, but to gain qualitative insights about the terms A and B, i.e., the insurance and crowding out effects.

We parameterize the model such that each period covers $J = 40$ actual years. We set $\alpha = 0.3$ and $\beta = 0.99^J$. With these parameters, the sufficient condition of dynamic efficiency in Proposition 5 is satisfied. Furthermore, $1 + g = (1 + 0.015)^J$, which is a standard value for the long run real productivity growth rate. Next, we set $\lambda = 0.1$ which assigns a relatively big role to social security—i.e., the pension period with weight $1 - \lambda$ is relatively long—and a small role to idiosyncratic risk—i.e., the working phase with weight λ is relatively short. We set the log variance of innovations of the idiosyncratic income process to an annual value of 0.01, corresponding to conventional estimates. Given the periodicity of $J = 40$ years, this means that $\sigma_\eta^2 = \exp(40 \cdot 0.01) - 1 \approx 0.5$. We vary the standard deviation of aggregate risk ($\sqrt{AR}$) from 0 to 1 to highlight how the results change in response.[18] We compute the expected values of all non-linear expressions by Gaussian Quadrature methods. We evaluate the integrals using $n_p = 5$ nodes.

Figure 1 displays the terms A and B, the total effect $A + B$ as well as the semi-

[17] Indeed, from a quantitative perspective it may be necessary to model both risks to find net welfare gains in general equilibrium, as we do in our quantitative analysis in Harenberg and Ludwig (2015).

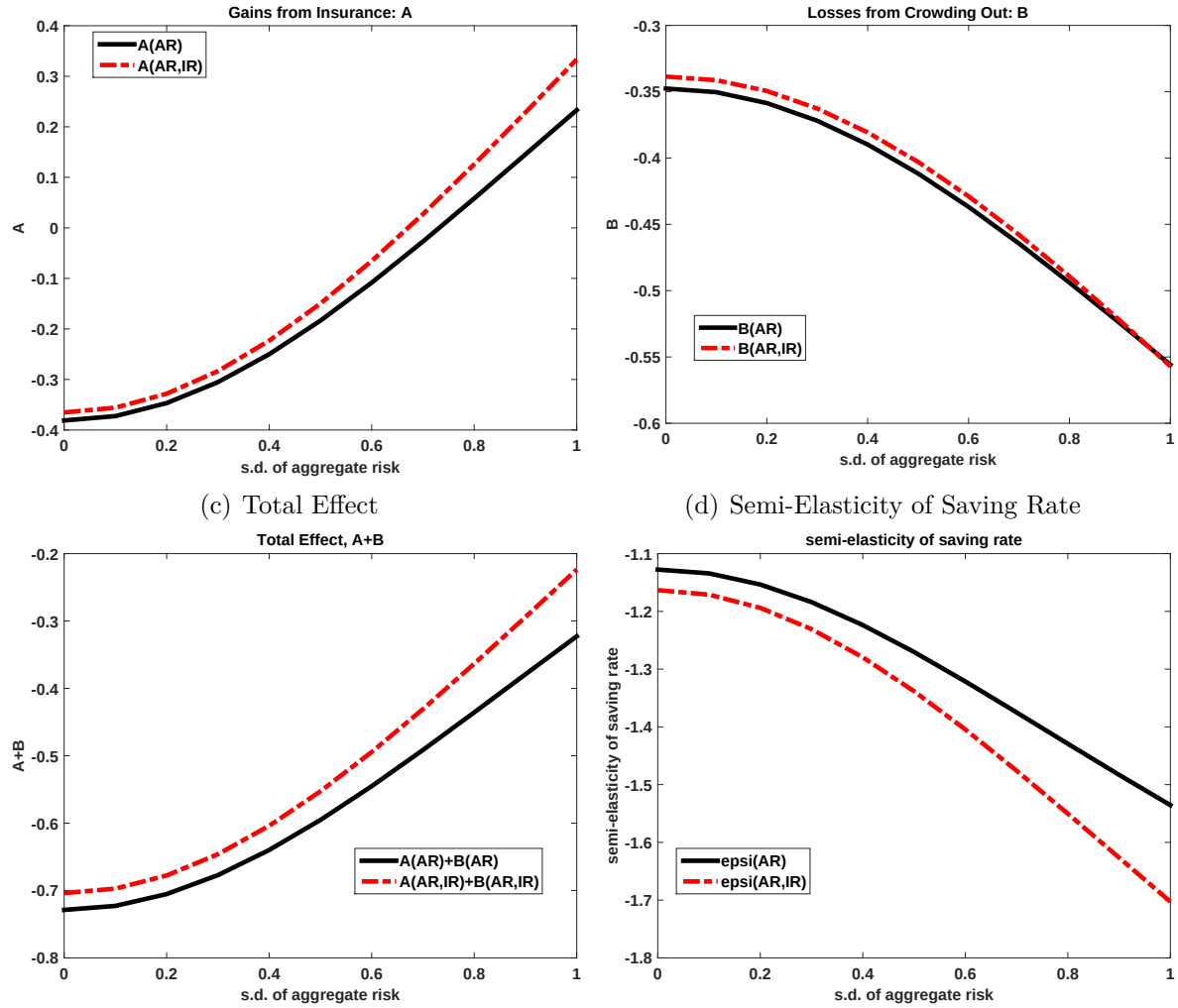
[18] Recall from Proposition 4 that the aggregate productivity shocks drop out in general equilibrium. Therefore, $\sqrt{AR} = \sigma_\varrho$.

elasticity of the saving rate $\epsilon_{s,\tau}$, as a function of $\sqrt{AR}$. In each panel, there is a black solid line representing an economy with only aggregate risk, and a red dash-dotted line representing an economy with aggregate and idiosyncratic risk.

Figure 1: Welfare Effects in General Equilibrium: High Discount Factor

(a) Term A

(b) Term B



(c) Total Effect

(d) Semi-Elasticity of Saving Rate

Notes: A and B are defined in proposition (4). Aggregate risk is $AR = \sigma_\varrho^2$, because σ_ζ does not enter in (4). $A(AR), B(AR)$ are for an economy with only aggregate risk, and $A(AR, IR), B(AR, IR)$ are for the economy with two separate risks.

With regard to term A, shown in panel (a), we see that the dash-dotted line, $A(AR, IR)$, lies above the solid line, $A(AR)$. This is not surprising, because social security is more beneficial in an economy with both risks, since it can insure against both. The lines start below zero, because the economy is dynamically efficient, but turn positive for a suffi-

cient amount of aggregate risk. We also see that the welfare effects are strictly increasing and convex in aggregate risk. The reason for this is that agents are risk-averse. The most important feature of this figure is that the distance between the two lines increases as we increase aggregate risk. This is due to the interaction between idiosyncratic and aggregate risk.

With regard to term B recall from Proposition 6 that it is ambiguous whether idiosyncratic risk should increase or decrease it. Likewise, we concluded that the interactions can increase or decrease B. In this calibration, the presence of idiosyncratic risk turns out to reduce the welfare costs from crowding out, i.e., the red dash-dotted line is above the black solid line. As aggregate risk increases, the gap becomes smaller, which means that the interactions increase the welfare costs from crowding out.

We see that the total effect, $A+B$, displayed in Panel (c) is increasing in aggregate risk. This is so because the marginal welfare benefits from insurance (term A) dominate the marginal welfare losses from crowing out (term B) for every unit of additional aggregate risk. More importantly, the gap between the two lines in Panel (c) increases, which means that the interactions in term A dominate those in term B. Importantly, this finding is for a low discounting scenario with $\beta = 0.99^J$. In our Supplementary Appendix, we also present results for high discounting (where we choose $\beta = 0.95^J$). In that case, crowding out dominates and idiosyncratic risk also increases its welfare costs. We therefore conclude that it is ultimately a quantitative question whether there are net benefits from social security and whether the interactions of risks in- or decrease those.

We also complement this analysis by plotting in Panel (d) the semi-elasticity $\epsilon_{s,\tau}|_{\tau=0}$. This confirms our earlier conjecture that it decreases in aggregate risk and idiosyncratic risk and that the cross partial derivative is negative as well (the gap between the two curves is increasing).

5 Conclusion

This paper develops an analytically tractable model with two overlapping generations where households are subject to aggregate business cycle and idiosyncratic productivity risk. We use this model to study the welfare consequences of introducing a marginal pay-as-you-go social security system. We highlight important indirect interactions between aggregate and idiosyncratic risks which are present although these risks are orthogonal by construction. These interactions measure how the variance of retirement consumption increases over-proportionally in the presence of both risks. Hence, the welfare gain from

insurance against both risks is greater than the sum of the gains from insurance against each risk. We first demonstrate this insurance channel in a partial equilibrium. Then, in general equilibrium, we oppose this insurance channel with the welfare loss from crowding out of capital which arises due to distortionary taxation.

Our central result here is that when idiosyncratic risk is increased, the welfare losses from crowding out are determined by two opposing forces. When social security insures larger risks, the crowding out of savings will be stronger, which represents a welfare loss. This is mitigated by the fact that higher idiosyncratic risk induces larger precautionary savings, which means that households profit more from the higher returns that result from the crowding out of aggregate capital. It remains ambiguous how the interactions affect these two forces. Thus, while the insurance gains of social security are unambiguously increased through the interactions, their impact on the welfare losses is ambiguous.

We therefore conclude that it is a quantitative question whether social security in- or decreases the net welfare effects in an economy with both risks. We address this quantitative question in our companion paper, Harenberg and Ludwig (2015). There we document that the effects of the interactions dominate on the side of the benefits. We also find that the introduction of a flat minimum pension is welfare improving once all household risks are appropriately taken into account thereby turning earlier findings in the literature upside down.

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A Appendix: Proofs

Proof of Proposition 1. Since $\tilde{\beta} = 1$, maximizing eq. (4) amounts to $max\mathbb{E}\, c_{i,2,t+1}^{1-\theta}$, where $c_{i,2,t+1}$ is given in (5). Increasing ex-ante utility for a marginal introduction of social security requires the first-order condition w.r.t. τ, evaluated at $\tau = 0$, to exceed zero, $\mathbb{E}\left[c_{i,2,t+1}^{-\theta}\frac{\partial c_{i,2,t+1}}{\partial \tau}\right]\Big|_{\tau=0} > 0$. We then get $\mathbb{E}\left[\frac{(1+g)\zeta_{t+1}(1+\lambda(1-\eta_{i,2,t+1}))-\zeta_t\bar{R}\varrho_{t+1}}{\left(\zeta_t\bar{R}\varrho_{t+1}+(1+g)\lambda\zeta_{t+1}\eta_{i,2,t+1}\right)^{\theta}}\right] > 0$, which gives equation (7). □

Proof of Proposition 2. Rewrite (8) as $A_{pe} = \mathbb{E}\left[\frac{aZ_1-bZ_2-1}{1+bZ_2}\right]$, where $a \equiv (1+\lambda)\frac{1+g}{\bar{R}}$, $b \equiv \lambda\frac{1+g}{\bar{R}}$, and $Z_1 \equiv \frac{\zeta_{t+1}}{\zeta_t\varrho_{t+1}}$, $Z_2 \equiv \frac{\zeta_{t+1}\eta_{i,2,t+1}}{\zeta_t\varrho_{t+1}}$. Take a second-order Taylor series approximation

around $Z_2 = Z_1 = 1$:

$$A_{pe} \approx \frac{1}{(1+b)^3}\left(ab^2\mathbb{E}[Z_1Z_2^2] - 3ab^2\mathbb{E}[Z_1Z_2] - ab\mathbb{E}[Z_1Z_2] + 3ab^2\mathbb{E}[Z_1] + \right.$$
$$\left. 3ab\mathbb{E}[Z_1] + a\mathbb{E}[Z_1] - (1+b)^3\right)$$

Observe that no interactions are present in term $\mathbb{E}Z_1$ and, by Assumption 1, there are also no interactions in term $\mathbb{E}[Z_1Z_2]$. However,

$$\mathbb{E}\left[Z_1Z_2^2\right] = \mathbb{E}\left[\frac{\zeta_{t+1}}{\zeta_t\varrho_{t+1}}\frac{\left(\zeta_{t+1}\eta_{2,i,t+1}\right)^2}{\left(\zeta_t\varrho_{t+1}\right)^2}\right] = \mathbb{E}\left[\zeta_{t+1}^3\right]\mathbb{E}\left[\frac{1}{\zeta_t^3}\right]\mathbb{E}\left[\eta_{2,i,t+1}^2\right]\mathbb{E}\left[\frac{1}{\varrho_{t+1}^3}\right],$$

so that an interaction between idiosyncratic and aggregate risk enters only through $\mathbb{E}[Z_1Z_2^2]$. We have $\mathbb{E}[\eta_2^2] = 1+\sigma_\eta^2$, $\mathbb{E}\left[\frac{1}{(\varrho)^3}\right] = \left(1+\sigma_\varrho^2\right)^6$ and $\mathbb{E}\zeta^3 = \left(1+\sigma_\zeta^2\right)^3$. Therefore $\mathbb{E}[Z_1Z_2^2] = \left(1+\sigma_\zeta^2\right)^9(1+\sigma_\varrho^2)^6(1+\sigma_\eta^2)$, from which the cross-derivatives follow. □

Proof of proposition 3. The proof is by guessing and verifying. As all households are ex-ante identical, we guess that

$$a_{2,t+1} = s(1-\tau)w_t = s(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha.$$

If this is correct, then the equilibrium dynamics are given by

$$K_{t+1} = a_{2,t+1} = s(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha$$

As $k_{t+1} = \frac{K_{t+1}}{\Upsilon_{t+1}(1+\lambda)}$ we get $k_{t+1} = \frac{1}{(1+g)(1+\lambda)}s(1-\tau)(1-\alpha)\zeta_t k_t^\alpha$.

To verify (9), notice that our guess for $a_{2,t+1}$ implies that

$$\begin{aligned} c_{1,t} &= (1-s)(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \\ c_{i,2,t+1} &= s(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1} + \\ &\quad + (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha\left(\lambda\eta_{i,2,t+1} + \tau\left(1+\lambda(1-\eta_{i,2,t+1})\right)\right), \end{aligned}$$

where we used the budget constraint. Employing (9) we get

$$c_{i,2,t+1} = \left(\alpha\varrho_{t+1}(1+\lambda) + (1-\alpha)\left(\lambda\eta_{i,2,t+1} + \tau\left(1+\lambda(1-\eta_{i,2,t+1})\right)\right)\right)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha.$$

Next, notice that the first-order-condition of household maximization gives

$$\begin{aligned}1 &= \beta\mathbb{E}_t\left[\frac{c_{1,t}(1+r_{t+1})}{c_{i,2,t+1}}\right]\\ &= \beta\mathbb{E}_t\left[\frac{c_{1,t}\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1}}{\left(\alpha\varrho_{t+1}(1+\lambda)+(1-\alpha)\left(\lambda\eta_{i,2,t+1}+\tau\left(1+\lambda(1-\eta_{i,2,t+1})\right)\right)\right)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^{\alpha}}\right]\\ &= \frac{\beta(1-s)}{s}\Phi,\end{aligned}$$

where Φ is defined in eq. (11). Equation (10) immediately follows. Since the problem is convex, the solution is unique. As for the upper bound on Φ observe that $\Phi = 1$ for $\lambda = 0$. For $\lambda > 0$, $\Phi = \mathbb{E}_t\left[\frac{1}{1+x}\right]$ for $x \equiv \frac{1-\alpha}{\alpha(1+\lambda)\varrho_{t+1}}\left(\lambda\eta_{i,2,t+1}+\tau\left(1+\lambda(1-\eta_{i,2,t+1})\right)\right)$. Our assumptions ensure that $x \geq 0$, hence $\Phi = \mathbb{E}_t\left[\frac{1}{1+x}\right] \leq 1$, which implies that $s \leq \frac{\beta}{1+\beta}$. □

Proof of Proposition 4. 1. Recursive substitution of eq. (9) gives

$$k_{t+1} = \left(\frac{1}{(1+g)(1+\lambda)}s(\tau)(1-\tau)(1-\alpha)\right)^{\frac{1-\alpha^{q+1}}{1-\alpha}}\left(\prod_{i=0}^{q}\zeta_{t-i}^{\alpha^i}\right)k_{t-q}^{\alpha^{1+q}} \tag{15}$$

for any initial capital stock k_{t-q}. For $q \to \infty$ we get

$$k_{t+1} = \left(\frac{1}{(1+g)(1+\lambda)}s(1-\tau)(1-\alpha)\right)^{\frac{1}{1-\alpha}}\left(\prod_{i=0}^{\infty}\zeta_{t-i}^{\alpha^i}\right) = k_{ms}\left(\prod_{i=0}^{\infty}\zeta_{t-i}^{\alpha^i}\right) = k_{ms}\,d(\zeta,t)$$

where k_{ms} denotes the capital stock that would obtain in equilibrium if nature would draw $\zeta_t = 1$ in all periods $t-q,\ldots,t$, for $q \to \infty$ (mean shock equilibrium).

2. Rewrite (2b) using the social security budget to make the excess return explicit

$$\begin{aligned}c_{i,2,t+1} = \Bigg(&s\zeta_t\varrho_{t+1}\bar{R}_{t+1} + \lambda\eta_{i,2,t+1}\frac{\bar{w}_{t+1}}{\bar{w}_t} + \\ &+\tau\left((1+\lambda(1-\eta_{i,2,t+1}))\frac{\bar{w}_{t+1}}{\bar{w}_t} - s\zeta_t\varrho_{t+1}\bar{R}_{t+1}\right)\Bigg)\bar{w}_t\zeta_{t+1},\end{aligned}$$

where $\bar{w}_t = \Upsilon_t(1-\alpha)k_t^{\alpha}$ and $\bar{R}_{t+1} = \alpha k_{t+1}^{\alpha-1}$. From step 1 we get that in the mean shock equilibrium $\bar{w}_t = \Upsilon_t(1-\alpha)k_{ms}^{\alpha}d(\zeta,t-1)^{\alpha}$, $\bar{R}_{t+1} = \alpha k_{ms}^{\alpha-1}d(\zeta,t)^{\alpha-1}$ and $\frac{\bar{w}_{t+1}}{\bar{w}_t} = (1+g)\left(\frac{d(\zeta,t)}{d(\zeta,t-1)}\right)^{\alpha}$. Noting that $\frac{d(\zeta,t)}{d(\zeta,t-1)^{\alpha}} = \zeta_t$, consumption in $j = 1,2$

can be written as:

$$c_{1,t} = (1-s)(1-\tau)\Upsilon_t\zeta_t(1-\alpha)k_{ms}^{\alpha}d(\zeta,t-1)^{\alpha} \equiv c_{1,t}(\tau,k_{ms},s) \tag{16}$$

$$c_{i,2,t+1} = \left(s\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \lambda\eta_{i,2,t+1}(1+g) + \tau\left((1+g)(1+\lambda(1-\eta_{i,2,t+1})) - s\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right) \cdot \Upsilon_t(1-\alpha)k_{ms}^{\alpha}\zeta_t\zeta_{t+1}d(\zeta,t-1)^{\alpha}d(\zeta,t)^{\alpha-1} \equiv c_{2,t+1}(\tau,k_{ms},s). \tag{17}$$

3. Using (16) and (17) in (4), we can write ex-ante utility as an indirect utility function

$$\mathbb{E}u_t = \mathbb{E}\left[u(c_{1,t}(\tau,k_{ms},s) + \beta\mathbb{E}_t\left[u(c_{2,t+1}(\tau,k_{ms},s)\right]\right] \tag{18}$$

where we use the law of iterated expectations to factor in the conditional expectations operator $\mathbb{E}_t$. Maximization of the above with respect to τ gives rise to the first-order condition[19]

$$\underbrace{\mathbb{E}\left[\frac{\partial u(c_{1,t})}{\partial c_{1,t}}\frac{\partial c_{1,t}}{\partial \tau} + \beta\mathbb{E}_t\left[\frac{\partial u(c_{2,t+1})}{\partial c_{2,t+1}}\frac{\partial c_{2,t+1}}{\partial \tau}\right]\right]}_{\equiv A = A_1 + A_2} \underbrace{\mathbb{E}\left[\frac{\partial k_{ms}}{\partial \tau}\left(\frac{\partial u(c_{1,t})}{\partial c_{1,t}}\frac{\partial c_{1,t}}{\partial k_{ms}} + \beta\mathbb{E}_t\left[\frac{\partial u(c_{2,t+1})}{\partial c_{2,t+1}}\frac{\partial c_{2,t+1}}{\partial k_{ms}}\right]\right)\right]}_{\equiv B = B_1 + B_2} \tag{19}$$

where A_1 (B_1), respectively A_2 (B_2), captures the effects on the period 1, respectively period 2, subutility function.

4. Using the explicit expressions for consumption in the two periods from (16) and (17) in the above, we get, evaluated at $\tau = 0$,

$$A_1 = \mathbb{E}\frac{\partial \ln(1-\tau)}{\partial \tau} = -\frac{1}{1-\tau}\Big|_{\tau=0} = -1,$$
$$B_1 = \alpha(1+\beta)\frac{\partial \ln k_{ms}}{\partial \tau}$$

and for A_2:

$$A_2 = \beta\mathbb{E}\left[\frac{(1+g)(1+\lambda(1-\eta_{i,2,t+1})) - s\varrho_{t+1}\alpha k_{ms}^{\alpha-1}}{s\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \lambda\eta_{i,2,t+1}(1+g)}\right] = \beta\mathbb{E}\left[\frac{\frac{1-\alpha}{\alpha(1+\lambda)}\frac{1+\lambda(1-\eta_{i,2,t+1})}{\varrho_{t+1}} - 1}{1 + \frac{(1-\alpha)\lambda}{\alpha(1+\lambda)}\frac{\eta_{i,2,t+1}}{\varrho_{t+1}}}\right].$$

[19] We use that, by the familiar envelope condition, $\mathbb{E}\left[\frac{\partial s}{\partial \tau}\left(\frac{\partial u(c_{1,t})}{\partial c_{1,t}}\frac{\partial c_{1,t}}{\partial s} + \beta\mathbb{E}_t\left[\frac{\partial u(c_{2,t+1})}{\partial c_{2,t+1}}\frac{\partial c_{2,t+1}}{\partial s}\right]\right)\right] = 0$ because $\frac{\partial c_{1,t}}{\partial s} = -1$ and $\frac{\partial c_{2,t+1}}{\partial s} = R_{t+1}$, hence the term in brackets is just the Euler equation.

Adding term A_1, which represents the effects of taxation on income, yields term A in the proposition. Furthermore, we get

$$B_2 \equiv \beta \mathbb{E}\left[\frac{\alpha \varrho_{t+1} s(\alpha-1) k_{ms}^{\alpha-2} \frac{\partial k_{ms}}{\partial \tau}}{s\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \lambda \eta_{i,2,t+1}(1+g)}\right] = \beta \mathbb{E}\left[-(1-\alpha)\frac{1}{1+\frac{(1-\alpha)\lambda}{\alpha(1+\lambda)}\frac{\eta_{i,2,t+1}}{\varrho_{t+1}}}\frac{\partial \ln k_{ms}}{\partial \tau}\right].$$

Adding B_1, yields

$$B = \left(\alpha(1+\beta) - \beta(1-\alpha)\,\Phi|_{\tau=0}\right)\mathbb{E}\left[\frac{\partial \ln k_{ms}}{\partial \tau}\right].$$

Turning to $\frac{\partial \ln k_{ms}}{\partial \tau}$ we find that, at $\tau = 0$, we have

$$\frac{\partial \ln k_{ms}}{\partial \tau} = \frac{1}{1-\alpha}\left(\frac{\partial \ln s}{\partial \tau} + \frac{\partial \ln(1-\tau)}{\partial \tau}\right) = -\frac{1}{1-\alpha}\left(1 - \epsilon_{s,\tau}|_{\tau=0}\right) < 0,$$

where the sign follows from the fact that the semi-elasticity of the saving rate in τ, $\epsilon_{s,\tau}|_{\tau=0}$, is negative. Precisely, it is given by

$$\epsilon_{s,\tau}|_{\tau=0} \equiv \frac{\partial s}{\partial \tau}\frac{1}{s}\Big|_{\tau=0} = -\beta(1-s)^2\frac{1}{s}\Psi\Big|_{\tau=0} = -\frac{\Psi|_{\tau=0}}{(1+\beta\,\Phi|_{\tau=0})\,\Phi|_{\tau=0}} < 0, \qquad (20)$$

where $\Psi|_{\tau=0} \equiv -\frac{\partial \Phi}{\partial \tau}\Big|_{\tau=0} = \mathbb{E}_t\left[\frac{\frac{1-\alpha}{\alpha(1+\lambda)}\frac{1+\lambda(1-\eta_{i,2,t+1})}{\varrho_{t+1}}}{\left(1+\frac{(1-\alpha)\lambda}{\alpha(1+\lambda)}\frac{\eta_{i,2,t+1}}{\varrho_{t+1}}\right)^2}\right] > 0$. Note that Φ is shown in (11) and $\Phi|_{\tau=0} > 0$. Term B in the proposition then follows.

□

Proof of Proposition 5 and Lemma 1. The aggregate resource constraint in the model is $c_{1,t} + c_{2,t} + K_{t+1} = F(K_t, \Upsilon_t L_t)$, where $c_{2,t} = \int c_{i,2,t} di$. By homogeneity of $F(\cdot,\cdot)$ maximizing per capita consumption $\bar{c} = \frac{c_{1,t}+c_{2,t}}{2}$ is equivalent to

$$\max\left\{\frac{F(K_t, \Upsilon_t L_t)}{N_t} - \frac{K_{t+1}}{N_t}\right\}. \qquad (21)$$

As $N_t = N_{t+1} = 2, L_t = 1+\lambda$ and recalling that $k_t = \frac{K_t}{\Upsilon_t L_t}$ we have that $\frac{K_{t+1}}{N_{t+1}} = k_{t+1}\Upsilon_{t+1}(1+\lambda)\frac{1}{2}$ and $\frac{L_t}{N_t} = (1+\lambda)\frac{1}{2}$. Maximizing (21) in steady state where $k_{t+1} = k_t = k$ is equivalent to $\max\{f(k) - (1+g)k\}$. Using that $f(k)) = k^\alpha$ we get the golden rule capital stock $k_{GR} = \left(\frac{\alpha}{1+g}\right)^{\frac{1}{1-\alpha}}$.

From equation (9) we get that the steady state capital stock in the deterministic $\lambda = 0$ economy is $k = \left(\frac{\beta(1-\alpha)}{(1+\beta)(1+g)}\right)^{\frac{1}{1-\alpha}}$. Hence the deterministic $\lambda = 0$ economy is dynamically efficient iff $\frac{\beta}{1+\beta} < \frac{\alpha}{1-\alpha}$.

Finally, observe that $B < 0$ iff $\alpha(1+\beta) - \beta(1-\alpha)\,\Phi|_{\tau=0} > 0$ which we can write as $\frac{\beta\,\Phi|_{\tau=0}}{1+\beta} < \frac{\alpha}{1-\alpha}$. Dynamic efficiency of the deterministic $\lambda = 0$ economy is a sufficient condition because from $0 < \Phi|_{\tau=0} \leq 1$, we have that $\frac{\beta\,\Phi|_{\tau=0}}{1+\beta} \leq \frac{\beta}{1+\beta} < \frac{\alpha}{1-\alpha}$, where the second inequality is condition (14). □

Proof of proposition 6. 1. The partial derivative of term A follows immediately from Proposition 2, by setting $\sigma_\zeta^2 = 0$ in $\frac{\partial\, A_{pe}|_{\theta=1}}{\partial \sigma_\eta^2}$.

2. The partial derivative of term B is given by

$$\frac{\partial B}{\partial \sigma_\eta^2} = \underbrace{\left(\alpha(1+\beta) - \beta(1-\alpha)\,\Phi|_{\tau=0}\right)}_{>0} \frac{1}{1-\alpha} \underbrace{\frac{\partial\, \epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2}}_{\lessgtr 0} + \beta \underbrace{\left(1 - \epsilon_{s,\tau}|_{\tau=0}\right)}_{>0} \underbrace{\frac{\partial\, \Phi|_{\tau=0}}{\partial \sigma_\eta^2}}_{>0}$$

where it remains to establish that, indeed, $\frac{\partial\,\Phi|_{\tau=0}}{\partial \sigma_\eta^2} > 0$ and $\frac{\partial\,\epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2} \lessgtr 0$:

(a) To evaluate $\frac{\partial\,\Phi|_{\tau=0}}{\partial \sigma_\eta^2}$, let $Z_3 = \frac{\eta_{i,2,t+1}}{\varrho_{t+1}}$. Take a second order Taylor-series approximation of $\Phi|_{\tau=0}$ around $Z_3 = 1$ to get

$$\Phi|_{\tau=0} \approx \left[\frac{b^2\,\mathbb{E}Z_3^2 - (3\,b^2 + b)\,\mathbb{E}Z_3 + 3\,b^2 + 3\,b + 1}{b^3 + 3\,b^2 + 3\,b + 1}\right],$$

where $b \equiv \frac{(1-\alpha)\lambda}{\alpha(1+\lambda)}$. With $\mathbb{E}\left[Z_3\right] = (1+\sigma_\varrho^2)$ and $\mathbb{E}\left[Z_3^2\right] = (1+\sigma_\eta^2)(1+\sigma_\varrho^2)^3$ get

$$\frac{\partial\,\Phi|_{\tau=0}}{\partial \sigma_\eta^2} = \frac{b^2\,\left(\sigma_\varrho^2 + 1\right)^3}{(1+b)^3} > 0 \tag{22}$$

and $\frac{\partial^2\,\Phi|_{\tau=0}}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} > 0$.

(b) For $\frac{\partial\,\epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2}$, we see from equation (20) that we need to determine $\frac{\partial\,\Psi|_{\tau=0}}{\partial \sigma_\eta^2}$. Let $Z_4 = \frac{1}{\varrho_{t+1}}$ and $a \equiv \frac{(1-\alpha)}{\alpha}$. Take a second-order Taylor series expansion of $\Psi|_{\tau=0}$ around $Z_3 = Z_4 = 1$ to get

$$\begin{aligned}\Psi|_{\tau=0} \approx & \frac{1}{(1+b)^4}\left[\left(3\,b^2\,a\,\mathbb{E}Z_3^2 - \left(8\,b^2 + 2\,b\right)\,a\,\mathbb{E}Z_3 + \left(6\,b^2 + 4\,b + 1\right)\,a\right)\,\mathbb{E}Z_4 + \right. \\ & \left.\left(2\,b^2 - b^3\right)\,\mathbb{E}Z_3^2 + \left(3\,b^3 - 4\,b^2 - b\right)\,\mathbb{E}Z_3 - 3\,b^3\right].\end{aligned}$$

To evaluate this expression under log-normality recall from above that $\mathbb{E}\left[Z_3^2\right] = (1+\sigma_\eta^2)(1+\sigma_\varrho^2)^3$ and observe that $\mathbb{E}\left[Z_4\right] = 1+\sigma_\varrho^2$, $\mathbb{E}\left[Z_3^2 Z_4\right] = (1+\sigma_\eta^2)(1+\sigma_\varrho^2)^6$, and $\mathbb{E}\left[Z_3 Z_4\right] = (1+\sigma_\varrho^2)^3$. Consequently,

$$\frac{\partial\, \Psi|_{\tau=0}}{\partial \sigma_\eta^2} = \left(\sigma_\varrho^2+1\right)^3 \frac{3\,a\,b^2\,\left(\sigma_\varrho^2+1\right)^3 - b^3\, + 2\,b^2}{(b+1)^4} > 0 \tag{23}$$

and $\frac{\partial^2 \Psi|_{\tau=0}}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} > 0$. The positive sign of $\frac{\partial \Psi|_{\tau=0}}{\partial \sigma_\eta^2}$ follows from the fact that

$$3\,a\,b^2\,\left(\sigma_\varrho^2+1\right)^3 - b^3\, + 2\,b^2 > 3\,a\,b^2 - b^3\, + 2\,b^2 > 0,$$

which results from $3\,a\,b^2 - b^3 + 2\,b^2 > 0 \Leftrightarrow (3-\alpha)(1+\lambda) > (1-\alpha)\lambda$, because $\alpha \in (0,1)$ and $\lambda \in (0,1)$. As $\frac{\partial\, \Phi|_{\tau=0}}{\partial \sigma_\eta^2} > 0$ and $\frac{\partial\, \Psi|_{\tau=0}}{\partial \sigma_\eta^2} > 0$ we have $\frac{\partial\, \epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2} \gtreqless 0$. As $\frac{\partial^2 \Phi|_{\tau=0}}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} > 0$ and $\frac{\partial^2 \Psi|_{\tau=0}}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} > 0$ we have that $\frac{\partial^2\, \epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2 \partial \sigma_\varrho^2} \gtreqless 0$.

We now characterize a lower bound on the semi-elasticity such that $\frac{\partial\, \epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2} < 0$. We get $\frac{\partial\, \epsilon_{s,\tau}|_{\tau=0}}{\partial \sigma_\eta^2} = \beta^{-1} s \Phi^{-2} \left(2\frac{\Psi}{\Phi}\frac{\partial \Phi}{\partial \sigma_\eta^2} - \frac{\partial \Psi}{\partial \sigma_\eta^2}\right)$, which is less than zero if— making use of equations (20), (22) and (23)— $\epsilon_{s,\tau}|_{\tau=0} > -\frac{3\,a\left(\sigma_\varrho^2+1\right)^3 - b + 2}{2(1+\beta\Phi)(1+b)}$.

□

B Supplementary Appendix: Additional Material

B.1 Comparison to Three-Generations Model

The purpose of this subsection is to illustrate the quasi-formal equivalence of the two-period model with a subperiod structure to a three-period model. To this aim, consider a setting like in Subsection 3.1. Households work in the first two periods of life and are retired in the third. There is no idiosyncratic risk in the first period. We again assume that households only care about consumption in retirement (third period). Third period consumption is then

$$\begin{aligned} c_{i,3,t+2} &= w_t(1-\tau)R_{t+1}R_{t+2} + w_{t+1}\eta_{i,2,t+1}(1-\tau)R_{t+2} + 2\tau w_{t+1} \\ &= \bar{w}_t \left(\zeta_t \bar{R}^2 \varrho_{t+1}\varrho_{t+2} + (1+g)\zeta_{t+1}\eta_{i,2,t+1}\bar{R}\varrho_{t+2} + \right. \\ &+ \qquad \left. \tau\left(2(1+g)^2\zeta_{t+2} - \left(\zeta_t \bar{R}^2 \varrho_{t+1}\varrho_{t+2} + (1+g)\zeta_{t+1}\eta_{i,2,t+1}\bar{R}\varrho_{t+2}\right)\right)\right). \end{aligned}$$

To interpret this in light of the analysis in subsection 3.1, again consider the case where $\tau = 0$ and $\bar{w}_t = 1$. Then

$$c_{i,3,t+2} = \bar{R}^2 \underbrace{\zeta_t \varrho_{t+1}\varrho_{t+2}}_{\equiv\psi} + (1+g)\bar{R} + (1+g)\bar{R}\left(\underbrace{\zeta_{t+1}\eta_{i,2,t+1}\varrho_{t+2}}_{\equiv\phi} - 1\right).$$

Observe that this is not a situation with mean zero independent "background" risk because of aggregate return risk in the second period: Term ϱ_{t+2} shows up in both random variables ψ and ϕ to the effect that ψ and ϕ are not independent.

Therefore, we additionally assume, somewhat artificially, that households only have access to a risk-free saving technology in the second period of life. We then get

$$c_{i,3,t+2} = \bar{R}^2 \underbrace{\zeta_t \varrho_{t+1}}_{\equiv\tilde{\psi}} + (1+g)\bar{R} + (1+g)\bar{R}\left(\underbrace{\zeta_{t+1}\eta_{i,2,t+1}}_{\equiv\tilde{\phi}} - 1\right).$$

This is formally equivalent to equation (6), i.e., we are back at a situation with independent additive "background" risk, again with the multiplicative interaction of risks via ζ_{t+1} in term $\tilde{\phi}$.

B.2 Welfare in General Equilibrium for $\lambda = 0$

This subsection derives the welfare effects in general equilibrium for the special case where $\lambda = 0$, because in this case, a very intuitive condition obtains. That is, we start from Proposition 4 and shut down all effects of idiosyncratic risk. To interpret term A, notice that for $\lambda = 0$ we can write

$$\frac{A}{1+\beta} = \frac{(1-\alpha)\beta}{\alpha(1+\beta)}\mathbb{E}\left[\frac{1}{\varrho_{t+1}}\right] - 1. \tag{24}$$

To further interpret this term, consider an artificial economy, namely the steady state of the mean-shock equilibrium (MSE) of the economy.

Definition 1. *In the mean-shock equilibrium, equilibrium dynamics are characterized by equation (9) but nature always draws the mean of aggregate shocks, i.e., $\zeta_t = \varrho_t = 1$ for all t.*

From this definition the steady state mean shock equilibrium capital stock in the $\lambda = 0$, $\tau = 0$ economy, denoted by k_{ms}, is given by

$$k_{ms} = \left(\frac{(1-\alpha)\beta}{(1+\beta)(1+g)}\right)^{\frac{1}{1-\alpha}}. \tag{25}$$

Consequently, the mean shock expected gross return is $R_{ms} = \alpha k_{ms}^{\alpha-1} = (1+g)\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}$. Hence, in this mean shock equilibrium, term $A > 0$ iff

$$\frac{1+g}{R_{ms}}\mathbb{E}\left[\frac{1}{\varrho_{t+1}}\right] - 1 > 0.$$

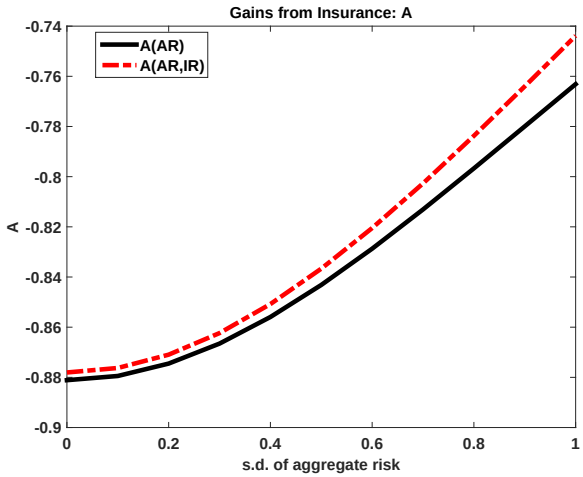
This means that the risk adjusted return of social security has to exceed the rental rate in the mean shock equilibrium. A mean preserving spread of ϱ_{t+1} increases A because insurance becomes more valuable. This establishes the analogy to our earlier interpretation of term $A_{pe}|_{\theta=1}$ in Subsection 3.1 for an economy without any idiosyncratic risk.

B.3 Numerical Results with High Discounting (low β)

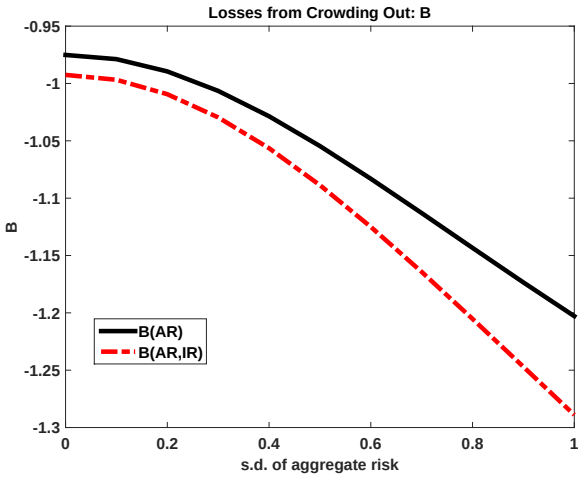
Figure 2 repeats our numerical exercise for $\beta = 0.95^J$. We now find that the welfare losses from crowding out dominate and the interactions increase the total effect negatively.

Figure 2: Welfare Effects in General Equilibrium: Low Discount Factor

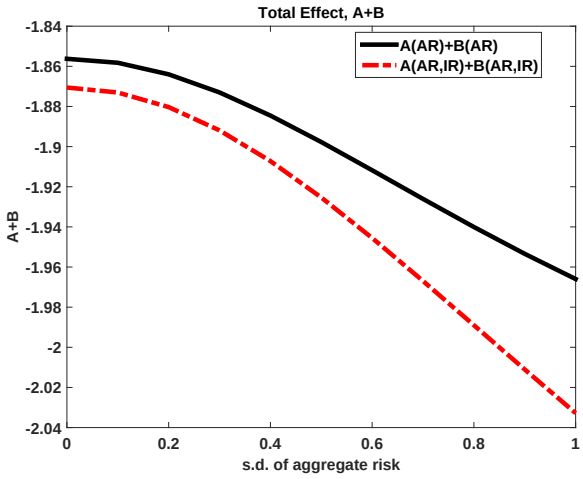
(a) Term A



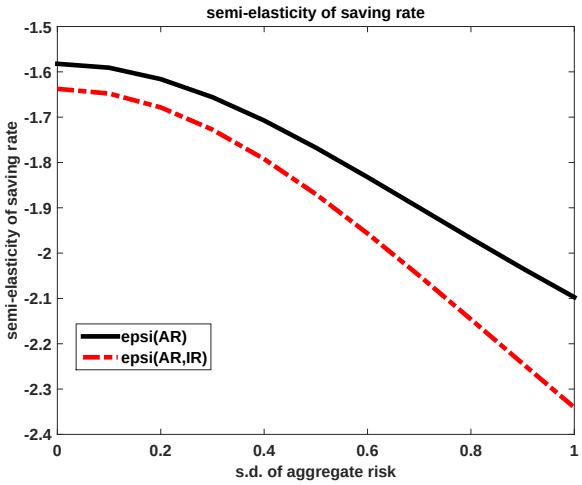
(b) Term B



(c) Total Effect



(d) Semi-Elasticity of Saving Rate



Notes: A and B are defined in proposition (4). Aggregate risk is $AR = \sigma_\varrho^2$, because σ_ζ does not enter in (4). $A(AR), B(AR)$ are for an economy with only aggregate risk, and $A(AR, IR), B(AR, IR)$ are for the economy with two separate risks.

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